Fatigue and Fracture

Multiaxial Fatigue

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When is Multiaxial Fatigue Important?

- Complex state of stress
- Complex out of phase loading
Uniaxial Stress

one principal stress
one direction
Proportional Biaxial

principal stresses vary proportionally but do not rotate

\[ \sigma_1 = \alpha \sigma_2 = \beta \sigma_3 \]
Nonproportional Multiaxial

Principal stresses may vary nonproportionally and/or change direction
Crankshaft
Shear and Normal Strains

\[
\begin{align*}
\gamma &= 0.005 \\
\varepsilon &= 0.0025
\end{align*}
\]
Shear and Normal Strains

\[ \varepsilon \]

\[ \gamma \]

45°

135°
3D stresses

Longitudinal Tensile Strain

Transverse Compression Strain

Thickness

50 mm

30 mm

15 mm

7 mm
Book

Multiaxial Fatigue

DARRELL F. SOCIE • GARY B. MARQUIS
Outline

- State of Stress
- Stress-Strain Relationships
- Fatigue Mechanisms
- Multiaxial Testing
- Stress Based Models
- Strain Based Models
- Fracture Mechanics Models
- Nonproportional Loading
- Stress Concentrations
State of Stress

- Stress components
- Common states of stress
- Shear stresses
Stress Components

Six stresses and six strains
Stresses Acting on a Plane
Principal Stresses

\[ \sigma^3 - \sigma^2(\sigma_X + \sigma_Y + \sigma_Z) + \sigma(\sigma_X\sigma_Y + \sigma_Y\sigma_Z\sigma_X\sigma_Z - \tau_{XY}^2 - \tau_{YZ}^2 - \tau_{XZ}^2) \]
\[ - (\sigma_X\sigma_Y\sigma_Z + 2\tau_{XY}\tau_{YZ}\tau_{XZ} - \sigma_X\tau_{YZ}^2 - \sigma_Y\tau_{ZX}^2 - \sigma_Z\tau_{XY}^2) = 0 \]
Stress and Strain Distributions

Stresses are nearly the same over a 10° range of angles.
Tension

\[ \sigma_2 = \sigma_3 = 0 \]

\[ \varepsilon_2 = \varepsilon_3 = -\nu \varepsilon_1 \]
Torsion

\[ \sigma_1 = \tau_{xy} \]

\[
\begin{align*}
\sigma_1 &= \tau_{xy} \\
\sigma_3 &= \tau_{xy} \\
\sigma_2 &= \tau_{xy}
\end{align*}
\]
Biaxial Tension

\[ \sigma_1 = \sigma_y \]

\[ \sigma_1 = \sigma_x \]

\[ \sigma_3 \]

\[ \varepsilon_3 = -\frac{2\nu}{1-\nu} \varepsilon \]
Shear Stresses

Maximum shear stress

\[ \tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2} \]

Mises:

\[ \bar{\sigma} = \frac{3}{\sqrt{2}} \tau_{\text{oct}} \]

Octahedral shear stress

\[ \tau_{\text{oct}} = \frac{1}{3} \sqrt{(\sigma_1 - \sigma_3)^2 + (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2} \]

\[ \tau_{\text{oct}} = \frac{3}{2\sqrt{2}} \tau_{13} = 0.94 \tau_{13} \]
Maximum and Octahedral Shear

- Maximum shear
- Octahedral shear

18% decrease

6% increase

\[ \sigma_3 / \sigma_1 \]

torsion, tension, biaxial tension
State of Stress Summary

- Stresses acting on a plane
- Principal stress
- Maximum shear stress
- Octahedral shear stress
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The Fatigue Process

- Crack nucleation
- Small crack growth in an elastic-plastic stress field
- Macroscopic crack growth in a nominally elastic stress field
- Final fracture
Mode I Growth
Mode II Growth

- shear stress
- slip bands
- crack growth direction (10 µm scale)
1045 Steel - Tension

![Graph showing Fatigue Life versus Damage Fraction](image)

- **Fatigue Life, $2N_f$**
- **Damage Fraction $N/N_f$**

- **Nucleation**
- **Tension**
- **Shear**

- **100 μm crack**

- **Inset images** showing material microstructure.
1045 Steel - Torsion

![Graph showing fatigue life and damage fraction for 1045 Steel in torsion.](image)

- **Damage Fraction $N/N_f$**
  - Y-axis: $0.0$, $0.2$, $0.4$, $0.6$, $0.8$, $1.0$
  - X-axis: $10^1$, $10^2$, $10^3$, $10^4$, $10^5$, $10^6$, $10^7$

- **Fatigue Life, $2N_f$**
  - Y-axis: $0$, $N_f$, $2N_f$
  - X-axis: $10^1$, $10^2$, $10^3$, $10^4$, $10^5$, $10^6$, $10^7$

- **Nucleation**
- **Tension**
- **Shear**
304 Stainless Steel - Torsion

![Graph showing damage fraction vs fatigue life for shear and tension loads.](image)

- **Damage Fraction** $N/N_f$
- **Fatigue Life, $2N_f$**

- **Nucleation**
- **Tension**
- **Shear**
304 Stainless Steel - Tension

![Graph showing the relationship between damage fraction and fatigue life for 304 stainless steel under tension. The graph plots damage fraction $N/N_f$ on the y-axis against fatigue life $2N_f$ on the x-axis, with nucleation and tension phases indicated.]
Inconel 718 - Torsion

Fatigue Life, $2N_f$

Damage Fraction $N/N_f$

Tension
Shear

Damage Fraction $N/N_f$

Fatigue Life, $2N_f$

Nucleation

0.1 mm
Inconel 718 - Tension

The graph shows the relationship between fatigue life and damage fraction for Inconel 718 under tension and shear conditions. The fatigue life is represented on a logarithmic scale, with the damage fraction N/N_f plotted against the fatigue life 2N_f.

- **Tension** curve shows a rapid increase in damage fraction with fatigue life.
- **Shear** curve indicates a more gradual increase in damage fraction.

The inset images provide visual representation of the fatigue processes, with nucleation visible in the shear condition.
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Fatigue Mechanisms Summary

- Fatigue cracks nucleate in shear
- Fatigue cracks grow in either shear or tension depending on material and state of stress
Stress Based Models

- Sines
- Findley
- Dang Van
Bending Torsion Correlation

Shear stress in torsion vs. 1/2 Bending fatigue limit

- Shear stress
- Octahedral stress
- Principal stress

Shear stress in bending vs. 1/2 Bending fatigue limit

0.5

1.0

2.0

0

0.5

1.0

1.5

2.0
Test Results

- Cyclic tension with static tension
- Cyclic torsion with static torsion
- Cyclic tension with static torsion
- Cyclic torsion with static tension
Cyclic Tension with Static Tension
Cyclic Torsion with Static Torsion

Shear Stress Amplitude

Shear Fatigue Strength

Maximum Shear Stress

Shear Yield Strength
Cyclic Tension with Static Torsion

![Graph showing cyclic tension with static torsion](image-url)
Cyclic Torsion with Static Tension

-1.5 -1.0 -0.5 0 0.5 1.0 1.5
Axial mean stress

1.5 1.0 0.5 0
Shear fatigue strength

Torsion shear stress

Yield strength
Conclusions

- Tension mean stress affects both tension and torsion
- Torsion mean stress does not affect tension or torsion
\[ \frac{\Delta \tau_{\text{oct}}}{2} + \alpha (3 \sigma_h) = \beta \]

\[ \frac{1}{6 \sqrt{\left( \Delta \sigma_x - \Delta \sigma_y \right)^2 + \left( \Delta \sigma_x - \Delta \sigma_z \right)^2 + \left( \Delta \sigma_y - \Delta \sigma_z \right)^2 + 6 \left( \Delta \tau_{xy} + \Delta \tau_{xz} + \Delta \tau_{yz} \right) + \alpha \left( \sigma_{x\text{mean}} + \sigma_{y\text{mean}} + \sigma_{z\text{mean}} \right) = \beta } \]
\[
\left( \frac{\Delta \tau}{2} + k \sigma_n \right)_{\text{max}} = f
\]
Bending Torsion Correlation

Shear stress in bending

1/2 Bending fatigue limit

Shear stress

Octahedral stress

Principal stress

Shear stress in torsion

1/2 Bending fatigue limit
\[ \tau(t) + a\sigma_h(t) = b \]

\[ \Sigma_{ij}(M,t) \ E_{ij}(M,t) \]
Isotropic Hardening

Failure occurs when the stress range is not elastic.
Multiaxial Kinematic and Isotropic

Yield domain expands and translates

ρ* stabilized residual stress
\( \tau(t) + a\sigma_h(t) = b \)

Dang Van (continued)
Stress Based Models Summary

Sines: \[ \frac{\Delta \tau_{oct}}{2} + \alpha(3\sigma_h) = \beta \]

Findley: \[ \left( \frac{\Delta \tau}{2} + k\sigma_n \right)_{\text{max}} = f \]

Dang Van: \[ \tau(t) + a\sigma_h(t) = b \]
Model Comparison $R = -1$

![Graph showing model comparison for $R = -1$. The graph compares Goodman, Findley, and Sines models for various stress ratios.](image-url)
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Strain Based Models

- Plastic Work
- Brown and Miller
- Fatemi and Socie
- Smith Watson and Topper
- Liu
Octahedral Shear Strain

![Graph showing the relationship between plastic octahedral shear strain range and cycles to failure for Torsion and Tension tests.](image)
Plastic Work

Plastic Work per Cycle, MJ/m$^3$

Fatigue Life, $N_f$

- Torsion
- Axial
- $0^\circ$
- $90^\circ$
- $180^\circ$
- $135^\circ$
- $45^\circ$
- $30^\circ$
Brown and Miller

![Graph showing fatigue life versus normal strain amplitude](image)

\[ \Delta \gamma = 0.03 \]

- Fatigue Life, Cycles
- Normal Strain Amplitude, \( \Delta \varepsilon_n \)
Case A and B

Case A

Growth along the surface

Case B

Growth into the surface
Brown and Miller (continued)

\[ \begin{align*}
\varepsilon_n & \quad \text{Uniaxial} \\
\gamma/2 & \quad \text{Shear} \\
\end{align*} \]
\[ \Delta \hat{\gamma} = \left( \Delta \gamma_{\text{max}}^{\alpha} + S \Delta \varepsilon_{n}^{\alpha} \right)^{\frac{1}{\alpha}} \]

\[ \frac{\Delta \gamma_{\text{max}}}{2} + S \Delta \varepsilon_{n} = A \frac{\sigma_{f}^{'} - 2\sigma_{n,\text{mean}}}{E} (2N_{f})^{b} + B \varepsilon_{f}^{'} (2N_{f})^{c} \]
Loading Histories

\[ \gamma / \sqrt{3} \]

\( \varepsilon \)

C

F

G

H

I

J
Crack Length Observations

Crack Length, mm

Cycles

Crack Length Observations

F-495  H-491
J-603  I-471
C-399  G-304
\[ \frac{\Delta \gamma}{2} \left( 1 + k \frac{\sigma_{n,\text{max}}}{\sigma_y} \right) = \frac{\tau'_f}{G} (2N_f)^{b_0} + \gamma'_f (2N_f)^{c_0} \]
Smith Watson Topper
\[ \sigma_n \frac{\Delta \varepsilon_1}{2} = \frac{\sigma_f^{'2}}{E} (2N_f)^{2b} + \sigma_f \varepsilon_f (2N_f)^{b+c} \]
Virtual strain energy for both mode I and mode II cracking

\[ \Delta W_I = (\Delta \sigma_n \Delta \varepsilon_n)_{\text{max}} + (\Delta \tau \Delta \gamma) \]

\[ \Delta W_I = 4\sigma_f\varepsilon_f(2N_f)^{b+c} + \frac{4\sigma_f^2}{E}(2N_f)^{2b} \]

\[ \Delta W_{II} = (\Delta \sigma_n \Delta \varepsilon_n) + (\Delta \tau \Delta \gamma)_{\text{max}} \]

\[ \Delta W_{II} = 4\tau_f\gamma_f(2N_f)^{b_{o+co}} + \frac{4\tau_f^2}{G}(2N_f)^{2bo} \]
Cyclic Torsion

Cyclic Shear Strain

Cyclic Tensile Strain

Shear Damage

Tensile Damage
Cyclic Torsion with Static Tension

- Cyclic Torsion
- Static Tension
- Cyclic Shear Strain
- Cyclic Tensile Strain
- Shear Damage
- Tensile Damage
Cyclic Torsion with Compression

- Cyclic Torsion
- Static Compression
- Cyclic Shear Strain
- Shear Damage
- Cyclic Tensile Strain
- Tensile Damage
Cyclic Torsion with Tension and Compression

- Cyclic Torsion
- Static Compression
- Hoop Tension
- Cyclic Shear Strain
- Cyclic Tensile Strain
- Shear Damage
- Tensile Damage
## Test Results

<table>
<thead>
<tr>
<th>Load Case</th>
<th>$\Delta \gamma / 2$</th>
<th>$\sigma_{\text{hoop}}$ MPa</th>
<th>$\sigma_{\text{axial}}$ MPa</th>
<th>$N_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torsion</td>
<td>0.0054</td>
<td>0</td>
<td>0</td>
<td>45,200</td>
</tr>
<tr>
<td>with tension</td>
<td>0.0054</td>
<td>0</td>
<td>450</td>
<td>10,300</td>
</tr>
<tr>
<td>with compression</td>
<td>0.0054</td>
<td>0</td>
<td>-500</td>
<td>50,000</td>
</tr>
<tr>
<td>with tension and compression</td>
<td>0.0054</td>
<td>450</td>
<td>-500</td>
<td>11,200</td>
</tr>
</tbody>
</table>
Conclusions

- All critical plane models correctly predict these results
- Hydrostatic stress models can not predict these results
Loading History
## Model Comparison

### Summary of calculated fatigue lives

<table>
<thead>
<tr>
<th>Model</th>
<th>Equation</th>
<th>Life</th>
</tr>
</thead>
<tbody>
<tr>
<td>Epsilon</td>
<td>6.5</td>
<td>14,060</td>
</tr>
<tr>
<td>Garud</td>
<td>6.7</td>
<td>5,210</td>
</tr>
<tr>
<td>Ellyin</td>
<td>6.17</td>
<td>4,450</td>
</tr>
<tr>
<td>Brown-Miller</td>
<td>6.22</td>
<td>3,980</td>
</tr>
<tr>
<td>SWT</td>
<td>6.24</td>
<td>9,930</td>
</tr>
<tr>
<td>Liu I</td>
<td>6.41</td>
<td>4,280</td>
</tr>
<tr>
<td>Liu II</td>
<td>6.42</td>
<td>5,420</td>
</tr>
<tr>
<td>Chu</td>
<td>6.37</td>
<td>3,040</td>
</tr>
<tr>
<td>Gamma</td>
<td></td>
<td>26,775</td>
</tr>
<tr>
<td>Fatemi-Socie</td>
<td>6.23</td>
<td>10,350</td>
</tr>
<tr>
<td>Glinka</td>
<td>6.39</td>
<td>33,220</td>
</tr>
</tbody>
</table>
Strain Based Models Summary

- Two separate models are needed, one for tensile growth and one for shear growth.
- Cyclic plasticity governs stress and strain ranges.
- Mean stress effects are a result of crack closure on the critical plane.
Separate Tensile and Shear Models

\[ \sigma_1 = \tau_{xy} \]

Inconel 1045 steel stainless steel
Cyclic Plasticity

\[ \Delta \varepsilon \]
\[ \Delta \gamma \]
\[ \Delta \varepsilon^p \]
\[ \Delta \gamma^p \]
\[ \Delta \varepsilon \Delta \sigma \]
\[ \Delta \gamma \Delta \tau \]
\[ \Delta \varepsilon^p \Delta \sigma \]
\[ \Delta \gamma^p \Delta \tau \]
Mean Stresses

\[ \Delta \varepsilon_{eq} = \frac{\sigma_f' - \sigma_{\text{mean}}}{E} (2N_f)^b + \varepsilon_f' (2N_f)^c \]

\[ \frac{\Delta \gamma_{\text{max}}}{2} + S \Delta \varepsilon_n = (1.3 + 0.7S) \frac{\sigma_f' - 2\sigma_n}{E} (2N_f)^b + (1.5 + 0.5S) \varepsilon_f' (2N_f)^c \]

\[ \frac{\Delta \gamma}{2} \left( 1 + k \frac{\sigma_{n,\text{max}}}{\sigma_y} \right) = \frac{\tau_f'}{G} (2N_f)^{bo} + \gamma_f' (2N_f)^{co} \]

\[ \sigma_n \frac{\Delta \varepsilon_1}{2} = \frac{\sigma_f' \Delta \varepsilon_1^2}{E} (2N_f)^{2b} + \sigma_f' \varepsilon_f' (2N_f)^{b+c} \]

\[ \Delta W_i = \left[ (\Delta \sigma_n \Delta \varepsilon_n)_{\text{max}} + (\Delta \tau \Delta \gamma) \right] \left( \frac{2}{1-R} \right) \]
Outline

- State of Stress
- Stress-Strain Relationships
- Fatigue Mechanisms
- Multiaxial Testing
- Stress Based Models
- Strain Based Models
- Fracture Mechanics Models
- Nonproportional Loading
- Stress Concentrations
Nonproportional Loading

- In and Out-of-phase loading
- Nonproportional cyclic hardening
- Variable amplitude
In and Out-of-Phase Loading

\[ \varepsilon_x = \varepsilon_o \sin(\omega t) \]
\[ \gamma_{xy} = (1+\nu)\varepsilon_o \sin(\omega t) \]

In-phase

\[ \varepsilon_x = \varepsilon_o \cos(\omega t) \]
\[ \gamma_{xy} = (1+\nu)\varepsilon_o \sin(\omega t) \]

Out-of-phase
In-Phase and Out-of-Phase
Loading Histories

out-of-phase  \( \gamma_{xy}/2 \)  

diamond  \( \gamma_{xy}/2 \)

square  \( \gamma_{xy}/2 \)  

cross  \( \gamma_{xy}/2 \)
Loading Histories

- **in-phase**
- **out-of-phase**
- **diamond**
- **square**
- **cross**
## Findley Model Results

<table>
<thead>
<tr>
<th>Configuration</th>
<th>(\Delta\tau/2) MPa</th>
<th>(\sigma_{n,max}) MPa</th>
<th>(\Delta\tau/2 + 0.3\sigma_{n,max})</th>
<th>(N/N_{ip})</th>
</tr>
</thead>
<tbody>
<tr>
<td>in-phase</td>
<td>353</td>
<td>250</td>
<td>428</td>
<td>1.0</td>
</tr>
<tr>
<td>90° out-of-phase</td>
<td>250</td>
<td>500</td>
<td>400</td>
<td>2.0</td>
</tr>
<tr>
<td>diamond</td>
<td>250</td>
<td>500</td>
<td>400</td>
<td>2.0</td>
</tr>
<tr>
<td>square</td>
<td>353</td>
<td>603</td>
<td>534</td>
<td>0.11</td>
</tr>
<tr>
<td>cross - tension cycle</td>
<td>250</td>
<td>250</td>
<td>325</td>
<td>16</td>
</tr>
<tr>
<td>cross - torsion cycle</td>
<td>250</td>
<td>0</td>
<td>250</td>
<td>216</td>
</tr>
</tbody>
</table>

![Diagram](image)
Nonproportional Hardening

\[ \varepsilon_x = \varepsilon_0 \sin(\omega t) \]

\[ \gamma_{xy} = (1+\nu)\varepsilon_0 \sin(\omega t) \]

**In-phase**

\[ \varepsilon_x = \varepsilon_0 \cos(\omega t) \]

\[ \gamma_{xy} = (1+\nu)\varepsilon_0 \sin(\omega t) \]

**Out-of-phase**
In-Phase

Axial

Shear
90° Out-of-Phase

Axial

Shear

-0.003 to 0.003

-600 to 600

-0.006 to 0.006

-300 to 300

0.006
Critical Plane

Proportional
- $N_f = 38,500$
- $N_f = 310,000$

Out-of-phase
- $N_f = 3,500$
- $N_f = 40,000$
Loading Histories

$\gamma / \sqrt{3}$

0 1 2 3 4 5 6 7 8 9 10 11 12 13
Stress-Strain Response
Stress-Strain Response (continued)
Nonproportional hardening results in lower fatigue lives

All tests have the same strain ranges
Nonproportional Example

Case A

\[ \sigma_x \]

Case B

\[ \sigma_x \]

Case C

\[ \sigma_x \]

Case D

\[ \sigma_x \]

\[ \sigma_y \]

\[ \sigma_y \]

\[ \sigma_y \]

\[ \sigma_y \]
Shear Stresses

Case A

Case B

Case C

Case D
Simple Variable Amplitude History
Stress-Strain on 0° Plane

\begin{align*}
\sigma_x & \quad -300 \\
\varepsilon_x & \quad 0.003 \\
\tau_{xy} & \quad -150 \\
\gamma_{xy} & \quad 0.005
\end{align*}
Stress-Strain on 30° and 60° Planes

- 30° plane
  - Stress ($\sigma_{30}$) vs. Strain ($\varepsilon_{30}$)

- 60° plane
  - Stress ($\sigma_{60}$) vs. Strain ($\varepsilon_{60}$)
Stress-Strain on 120° and 150° Planes

120° plane

150° plane

\[ \sigma_{120} \] vs. \[ \varepsilon_{120} \]

\[ \sigma_{150} \] vs. \[ \varepsilon_{150} \]
Shear Strain History on Critical Plane
Fatigue Calculations

Load or strain history

Cyclic plasticity model

Stress and strain tensor

Search for critical plane
An Example

- Analysis model
  - Single event
  - 16 input channels
  - 2240 elements

Biaxial and Uniaxial Solution

Element rank based on critical plane damage

- Uniaxial solution
  Signed principal stress
- Critical plane solution

Damage

Element rank based on critical plane damage

10^{-2} to 10^{-8}
Nonproportional Loading Summary

- Nonproportional cyclic hardening increases stress levels
- Critical plane models are used to assess fatigue damage
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Notches

- Stress and strain concentrations
- Nonproportional loading and stressing
- Fatigue notch factors
- Cracks at notches
Notched Shaft Loading
Stress Concentration Factors

![Diagram showing stress concentration factors for bending and torsion with varying notch root radius and D/d ratio.](image)

- **Stress Concentration Factor**
  - **Bending**
  - **Torsion**

- **Notch Root Radius, ρ/d**
  - 0.025
  - 0.050
  - 0.075
  - 0.100
  - 0.125

- **D/d**
  - 2.20
  - 1.20
  - 1.04
Hole in a Plate
Stresses at the Hole

Stress concentration factor depends on type of loading
Shear Stresses during Torsion

\[ \tau_{r\theta} \]

\[ \sigma \]

\[ \frac{r}{a} \]

\[ 0 \]

\[ 0.5 \]

\[ 1.0 \]

\[ 1.5 \]
Torsion Experiments
Multiaxial Loading

- Uniaxial loading that produces multiaxial stresses at notches
- Multiaxial loading that produces uniaxial stresses at notches
- Multiaxial loading that produces multiaxial stresses at notches
Thickness Effects

The diagram illustrates the effect of thickness on longitudinal tensile strain and transverse compression strain. The thickness values are 50, 30, 15, and 7 mm. The x, y, and z axes are indicated, with the x-axis pointing to the right, the y-axis pointing upwards, and the z-axis pointing to the left. The thickness values are shown alongside the corresponding lines on the graph.
Multiaxial Loading

- Uniaxial loading that produces multiaxial stresses at notches
- Multiaxial loading that produces uniaxial stresses at notches
- Multiaxial loading that produces multiaxial stresses at notches
Applied Bending Moments
Bending Moments on the Shaft

Location

A

B

C

D

A'

B'

C'

D'

$M_Y$

$M_X$

Bending Moments on the Shaft
# Bending Moments

\[
\Delta M = \sum \Delta M^5
\]

<table>
<thead>
<tr>
<th>$\Delta M$</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
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<tbody>
<tr>
<td>2.82</td>
<td></td>
<td>1</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>2.00</td>
<td></td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.41</td>
<td></td>
<td>2</td>
<td></td>
<td>1</td>
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<tr>
<td>1.00</td>
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<td>2</td>
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<td>0.71</td>
<td></td>
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<td>2</td>
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</table>

<table>
<thead>
<tr>
<th>$\Delta M$</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.49</td>
<td>2.85</td>
<td>2.31</td>
<td>2.84</td>
<td></td>
</tr>
</tbody>
</table>
Combined Loading

\[
\sigma_1 = 1.72\sigma
\]

\[
\sigma_2 = -0.72\sigma
\]

\[
\lambda = -0.41
\]
Maximum Tensile Stress Location

\[ \sigma_1 = \sigma \]

\[ K_t = 3 \]

\[ \sigma_1 = 1.72\sigma \]

\[ K_t = 3.41 \]

\[ \sigma_1 = \tau \]

\[ K_t = 4 \]
In and Out of Phase Loading

In-phase

Out-of-phase

$K_t = 3$

$K_t = 4$

Damage location changes with load phasing
Multiaxial Loading

- Uniaxial loading that produces multiaxial stresses at notches
- Multiaxial loading that produces uniaxial stresses at notches
- Multiaxial loading that produces multiaxial stresses at notches
Out-of-phase shear loading is needed to produce nonproportional stressing
Fatigue Notch Factors

Stress concentration factor

Notch root radius, $\rho$

$K_t$ Bending

$K_t$ Torsion

$K_f$ Bending

$K_f$ Torsion

$D = 2.2$

$d = 2.2$
Fatigue Notch Factors (continued)

Peterson’s Equation

\[ K_f = 1 + \frac{K_T - 1}{1 + \frac{a}{r}} \]
Fracture Surfaces in Torsion

Circumferencial Notch

Shoulder Fillet
Neuber’s Rule

\[ K_t S K_t e = \sigma \varepsilon \]

Stress calculated with elastic assumptions

\[ ^e S ^e e = \sigma \varepsilon \]

For cyclic loading

\[ \Delta ^e S^2 = E \Delta \sigma \Delta \varepsilon \]
Multiaxial Neuber’s Rule

Define Neuber’s rule in equivalent variables

\[ \Delta^e S^2 = E \Delta \bar{\sigma} \Delta \bar{\varepsilon} \]

Stress strain curve

\[ \Delta \bar{\varepsilon} = \frac{\Delta \bar{\sigma}}{E} + \left( \frac{\Delta \bar{\sigma}}{K'} \right)^{\frac{1}{n'}} \]

Constitutive equation

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
= f(E,K',n')
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

Five equations and six unknowns
Ignore Plasticity Theory

\[ \varepsilon_2 = \frac{e\varepsilon_2}{e\varepsilon_1} \varepsilon_1 \]

\[ \varepsilon_3 = \frac{e\varepsilon_3}{e\varepsilon_1} \varepsilon_1 \]

\[ \sigma_2 = \frac{eS_2}{eS_1} \sigma_1 \]

\[ \sigma_3 = \frac{eS_3}{eS_1} \sigma_1 \]
\[ \frac{\sigma_2}{\sigma_1} = \frac{eS_2}{eS_1} \]

\[ \frac{\varepsilon_2}{\varepsilon_1} = \frac{e\varepsilon_2}{e\varepsilon_1} \]
Strain energy density

\[ \frac{\Delta \sigma_{ij} \Delta \varepsilon_{ij}}{\sum \Delta \sigma_{ij} \Delta \varepsilon_{ij}} = \frac{\Delta^e S_{ij} \Delta^e e_{ij}}{\sum \Delta^e S_{ij} \Delta^e e_{ij}} \]
Koettgen-Barkey-Socie

Structural Yield Surface

\[ \varepsilon = f_o(E_o, K_o, n_o) \]

Material Yield Surface

\[ \sigma = f(E, K', n') \]
Stress Intensity Factors

\[
\frac{da}{dN} = C (\Delta K_{eq})^m \quad K_I = F \Delta \sigma \sqrt{\pi a}
\]
Crack Growth From a Hole

![Graph showing crack growth from a hole with different values of lambda (\(\lambda\)) for various cycles and crack lengths.](image-url)
Notches Summary

- Uniaxial loading can produce multiaxial stresses at notches
- Multiaxial loading can produce uniaxial stresses at notches
- Multiaxial stresses are not very important in thin plate and shell structures
- Multiaxial stresses are not very important in crack growth
Multiaxial Fatigue