Probabilistic Aspects of Fatigue

Professor Darrell F. Socie

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Probabilistic Aspects of Fatigue

- Introduction
- Statistical Techniques
- Sources of Variability
Probabilistic Models

- Probabilistic models are no better than the underlying deterministic models
- They require more work to implement
- Why use them?
Quality and Cost

- Taguchi
  - Identify factors that influence performance
  - Robust design – reduce sensitivity to noise
  - Assess economic impact of variation

- Risk / Reliability
  - What is the increased risk from reduced testing?
Risk

Acceptable risk

Time, Flights etc
Reliability

Expected Failures

Fatigue Life

99 %
80 %
50 %
10 %
1 %
0.1 %

Expected Failures

Fatigue Life

10^3 10^4 10^5 10^6
Risk Contribution Factors

- Analysis Uncertainty
- Manufacturing Flaws
- Operating Temperature
- Material Properties
- Speed
Uncertainty and Variability

customers ← Stress → usage

Fatigue Life, $2N_f$

Strain Amplitude

failures

50%

100%

Strength

materials ← Strength → manufacturing

Strength
Deterministic – from past measurements the future position of a satellite can be predicted with reasonable accuracy.

Random – from past measurements the future position of a car can only be described in terms of probability and statistical averages.
Variability and uncertainty is accommodated by introducing safety factors. Larger safety factors are better, but how much better and at what cost?
Probabilistic Design

Reliability = 1 – P( Stress > Strength )
3σ Approach

3σ contains 99.87% of the data

\[ P( s < S ) = 2.3 \times 10^{-3} \]

If we use 3σ on both stress and strength

\[ P( \text{failure} ) = P( \Sigma \geq s \cap s \leq S ) = 5.3 \times 10^{-6} \approx 4.5\sigma \]

The probability of the part with the lowest strength having the highest stress is very small

For 3 variables, each at 3σ:

\[ P(\text{failure}) = 1.2 \times 10^{-8} \approx 5.7\sigma \]
Benefits

- Reduces conservatism (cost) compared to assuming the “worst case” for every design variable
- Quantifies life drivers – what are the most important variables and how well are they known or controlled?
- Quantifies risk
Reliability Analysis

Stressing Variables

Strength Variables

Analysis

P( Failure )
Probabilistic Analysis Methods

- Monte Carlo
  - Simple
  - Hypercube sampling
  - Importance sampling

- Analytical
  - First order reliability method FORM
  - Second order reliability method SORM
Statistical Techniques

- Normal Distributions
- LogNormal Distributions
- Monte Carlo
- Distribution Fitting
Let $\Sigma$ be the stress and $S$ the fatigue strength. Given the distributions of $\Sigma$ and $S$ find the probability of failure

$$P(\Sigma \geq s \cap s \leq S)$$
Normal Variables

Linear Response Function

\[ Z = a_0 + \sum_{i=1}^{n} a_i X_i \]

\[ X_i \sim N(\mu_i, C_i) \]

\[ \mu_z = a_0 + \sum_{i=1}^{n} a_i \mu_i \]

\[ \sigma_z = \sqrt{\sum_{i=1}^{n} a_i^2 \sigma_i^2} \]
Calculations

Let $Z$ be a random variable:

$$Z = S - \Sigma$$

$$\mu_Z = \mu_S - \mu_\Sigma$$

$$\mu_Z = 200 - 100 = 100$$

$$\sigma_Z = \sqrt{\sigma_S^2 + \sigma_\Sigma^2}$$

$$\sigma_Z = \sqrt{20^2 + 20^2} = 28.2$$
Failure Probability

\[ Z = S - \sum \]

Failure will occur whenever \( Z \leq 0 \)

\[ Z = \mu_z - z \sigma_z = 0 \]

\[ z = \frac{\mu_z}{\sigma_z} = \frac{100}{28.2} \]

\( z = 3.54 \) standard deviations

\[ P(\text{failure}) = 2 \times 10^{-4} \]

For this case only, a safety factor of 2 means a probability of failure of \( 2 \times 10^{-4} \). Other situations will require different safety factors to achieve the same reliability.
Failure Distribution

Stress, $\Sigma$  \quad Strength, $S$

What is the expected distribution in fatigue lives?
Fatigue Data

\[
\sigma_f' = \frac{\Delta S}{2 (2N_f)^b}
\]

\[
2N_f = \left( \frac{\Delta S}{2 \sigma_f'} \right)^{\frac{1}{b}}
\]
LogNormal Variables

\[ Z = a_o \prod_{i=1}^{n} X_i^{a_i} \]

a’s are constant and \( X_i \sim \text{LN}(x_i, C_i) \)

\[ \text{median } \bar{Z} = a_o \prod_{i=1}^{n} \bar{X}_i^{a_i} \]

\[ \text{COV } C_Z = \sqrt{\prod_{i=1}^{n} \left(1 + C_{X_i}^2 \right)^{a_i^2}} - 1 \]
Calculations

\[ \begin{align*}
\sigma_f' & \sim \text{LN}(1000, 0.1) \quad \sigma = 100 \\
\frac{\Delta S}{2} & \sim \text{LN}(250, 0.2) \quad \sigma = 50 \\
b & = -0.125 \\
2N_f &= \left( \frac{\Delta S}{2\sigma_f} \right)^{\frac{1}{b}} \\
Z &= 2N_f = \left( \frac{\Delta S}{2} \right)^{-8} \sigma_f^8 \\
\bar{Z} &= 2N_f = \left( \frac{\Delta \bar{S}}{2} \right)^{-8} \frac{1}{\sigma_f^8} \\
\text{COV}_Z &= \sqrt{\left(1 + \text{COV}_{\Delta S}^2\right)^{-8} \left(1 + \text{COV}_{\sigma_f}^2\right)^{8^2} - 1}
\end{align*} \]
Results

<table>
<thead>
<tr>
<th></th>
<th>$\Delta S/2$</th>
<th>$\sigma_f$</th>
<th>$2N_f$</th>
<th>Percentile</th>
<th>Life</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_x$</td>
<td>250</td>
<td>1000</td>
<td>355,368</td>
<td>99.9</td>
<td>17,706,069</td>
</tr>
<tr>
<td>COV$_x$</td>
<td>0.2</td>
<td>0.1</td>
<td>4.72</td>
<td>99</td>
<td>4,566,613</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>95</td>
<td>1,363,200</td>
</tr>
<tr>
<td>$\mu_{lnx}$</td>
<td>5.50</td>
<td>6.90</td>
<td>11.21</td>
<td>90</td>
<td>715,589</td>
</tr>
<tr>
<td>$\chi$</td>
<td>245</td>
<td>995</td>
<td>73,676</td>
<td>50</td>
<td>73,676</td>
</tr>
<tr>
<td>$\sigma_x$</td>
<td>50</td>
<td>100</td>
<td>1,676,831</td>
<td>10</td>
<td>7,586</td>
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<tr>
<td>$\sigma_{lnx}$</td>
<td>0.198</td>
<td>0.100</td>
<td>1.774</td>
<td>5</td>
<td>3,982</td>
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<tr>
<td>b =</td>
<td>-0.125</td>
<td></td>
<td></td>
<td>1</td>
<td>1,189</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.1</td>
<td>307</td>
</tr>
</tbody>
</table>
Monte Carlo Methods

\[
\frac{K_f \Delta S}{2} = \sqrt{E \left( \frac{\sigma_f'}{E} (2N_f)^{2b} + \sigma_f' \varepsilon_f' (2N_f)^{b+c} \right)}
\]

Given random variables for \( K_f \), \( \Delta S \), \( \sigma_f' \) and \( \varepsilon_f' \)
Find the distribution of \( 2N_f \)

\[ Z = 2N_f = ? \]
Simple Example

Probability of rolling a 3 on a die

\[ f_x(x) \]

1 2 3 4 5 6

--- 1/6

Uniform discrete distribution
Computer Simulation

1. Generate \( n \) random numbers between 1 and 6, all integers

2. Count the number of 3’s

Let \( X_i = 1 \) if 3

0 otherwise

\[
P(3) = \frac{1}{n} \sum_{i=1}^{n} X_i
\]
EXCEL

\[ =\text{ROUNDUP}(6 \times \text{RAND}(), 0) \]

\[ =\text{IF}(A1 = 3, 1, 0) \]

\[ =\text{SUM}($B$1:B1)/\text{ROW}(B1) \]

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0.333333</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0.25</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0.166667</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.142857</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.333333</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0.3</td>
</tr>
</tbody>
</table>
Results
Evaluate $\pi$

$$P(\text{inside circle}) \quad P = \frac{\pi r^2}{4}$$

$$\pi = 4 \times P$$

$$x = 2 \times \text{RAND()} - 1$$

$$y = 2 \times \text{RAND()} - 1$$

$$\text{IF}(x^2 + y^2 < 1, 1, 0)$$
Monte Carlo Simulation

Randomly choose values of $S$ and $\sigma'$ from their distributions

Repeat many times

\[ 2N_f = \left( \frac{\Delta S}{2\sigma_f} \right)^{\frac{1}{b}} \]
Randomly choose a value between 0 and 1

\[ x = F_x^{-1}( \text{RAND} ) \]
Generating Distributions in EXCEL

Normal

\[ = \text{NORMINV} (\text{RAND}(), \mu, \sigma) \]

Log Normal

\[ = \text{LOGINV} (\text{RAND}(), \ln \mu, \ln \sigma) \]
<table>
<thead>
<tr>
<th>$\sigma_f$</th>
<th>$\frac{\Delta S}{2}$</th>
<th>$2N_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>893</td>
<td>204</td>
<td>134,677</td>
</tr>
<tr>
<td>1102</td>
<td>301</td>
<td>32,180</td>
</tr>
<tr>
<td>852</td>
<td>285</td>
<td>6,355</td>
</tr>
<tr>
<td>963</td>
<td>173</td>
<td>929,249</td>
</tr>
<tr>
<td>1050</td>
<td>283</td>
<td>35,565</td>
</tr>
<tr>
<td>1080</td>
<td>265</td>
<td>77,057</td>
</tr>
<tr>
<td>965</td>
<td>313</td>
<td>8,227</td>
</tr>
<tr>
<td>1073</td>
<td>213</td>
<td>420,456</td>
</tr>
<tr>
<td>1052</td>
<td>226</td>
<td>224,000</td>
</tr>
<tr>
<td>954</td>
<td>322</td>
<td>5,878</td>
</tr>
<tr>
<td>965</td>
<td>240</td>
<td>68,671</td>
</tr>
<tr>
<td>993</td>
<td>207</td>
<td>277,192</td>
</tr>
<tr>
<td>1191</td>
<td>368</td>
<td>11,967</td>
</tr>
<tr>
<td>831</td>
<td>210</td>
<td>59,473</td>
</tr>
</tbody>
</table>
Simulation Results

<table>
<thead>
<tr>
<th>Cumulative Probability</th>
<th>Monte Carlo</th>
<th>Analytical</th>
</tr>
</thead>
<tbody>
<tr>
<td>99.9 %</td>
<td>11.25</td>
<td>11.21</td>
</tr>
<tr>
<td>99 %</td>
<td>1.79</td>
<td>1.77</td>
</tr>
</tbody>
</table>

Mean 11.25 11.21
Std 1.79 1.77
Summary

Simulation is relatively straightforward and simple

Obtaining the necessary input data and distributions is difficult
Maximum Load Data

Maximum force from 42 drivers

Median 431
COV 0.34
90% confidence COV 0.41
Maximum Load Data

Uncertainty in Variance is just as important, perhaps more important than the choice of the distribution.

Normal  COV = 0.34
LogNormal
Normal  COV = 0.41
Choose the “Best” Distribution

15 samples from a Normal Distribution

- LogNormal
- Weibull
- Gumble
- Normal
Distributions

- Normal
  - Strength
  - Dimensions

- LogNormal
  - Fatigue Lives
  - Large variance in properties or loads

- Gumble
  - Maximums in a population

- Weibull
  - Fatigue Lives
Central Limit Theorem

If $X_1, X_2, X_3 \ldots \ldots X_n$ is a random sample from the population, with sample mean $\bar{X}$, then the limiting form of

$$Z = \frac{\bar{X} - \mu_x}{\sigma / \sqrt{n}}$$

as $n \to \infty$ is the standard normal distribution
When there are many variables affecting the outcome, The final result will be normally distributed even if the individual variable distributions are not.

As a result, normal distributions are frequently assumed for all of the input variables.
Example

Probability of rolling a die

\[ f(x) \]

Uniform discrete distribution

Let \( Z \) be the summation of six dice

\[ Z = X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \]
Results

Central limit theorem states that the result should be normal for large n

500 trials

$\bar{X}_Z = 21.12$

$C_Z = 0.20$
Central Limit Theorem

Sums:  \( Z = X_1 \pm X_2 \pm X_3 \pm X_4 \pm \ldots \pm X_n \)

\( Z \to \) Normal as \( n \) increases

Products:  \( Z = X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot \ldots \cdot X_n \)

\( Z \to \) LogNormal as \( n \) increases

Normal and LogNormal distributions are often employed for analysis even though the underlying population distribution is unknown.
Key Points

- All variables are random and can be characterized by a statistical distribution with a mean and variance.
- The final result will be normally distributed even if the individual variable distributions are not.
Sources of Variability

- Customers
- Stress
- Usage
- Materials
- Strength
- Manufacturing

Stress, $\Sigma$

Strength, $S$

Fatigue Life, $2N_f$

Strain Amplitude

Fatigue Life vs Strain Amplitude
Variability and Uncertainty

Variability: Every apple on a tree has a different mass.
Uncertainty: The variety of the apple is unknown.

Variability: Fracture toughness of a material
Uncertainty: The correct stress intensity factor solution
Sources of Variability

- Stress Variables
  - Loading
  - Customer Usage
  - Environment

- Strength Variables
  - Material
  - Processing
  - Manufacturing Tolerance
  - Environment
Sources of Uncertainty

- Statistical Uncertainty
  - Incomplete data (small sample sizes)
- Modeling Error
  - Analysis assumptions
- Human Error
  - Calculation errors
  - Judgment errors
Modeling Variability

Central Limit Theorem:

Products: \[ Z = X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot \ldots \cdot X_n \]

\[ Z \rightarrow \text{LogNormal as } n \text{ increases} \]

\[ \sigma_{\ln X} \sim \text{COV}_X \]

\[ \text{COV}_X \text{ is a good measure of variability} \]
COV and LogNormal Distributions

<table>
<thead>
<tr>
<th>COV&lt;sub&gt;x&lt;/sub&gt;</th>
<th>Standard Deviation, ln x</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>68.3%</td>
</tr>
<tr>
<td>0.05</td>
<td>1.05</td>
</tr>
<tr>
<td>0.1</td>
<td>1.10</td>
</tr>
<tr>
<td>0.25</td>
<td>1.28</td>
</tr>
<tr>
<td>0.5</td>
<td>1.60</td>
</tr>
<tr>
<td>1</td>
<td>2.30</td>
</tr>
</tbody>
</table>

99.7% of the data is within a factor of ±1.33 of the mean for a COV = 0.1
Variability in Service Loading

- Quantifying Loading Variability
  - Maximum Load
  - Load Range
  - Equivalent Stress
Maximum Force

Maximum force from 42 automobile drivers

Median 431
COV 0.34
Maximum Load Correlation

Fatigue Lives

Maximum Load
Loading Variability

Cumulative Cycles

Load Range

54 Tractors / Drivers

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Variability in Loading

54 Tractors/Drivers
COV  0.53

Cumulative Probability

Equivalent Load

\[ \Delta S_{eq} = \sqrt[n]{\sum \Delta S^n} \]
Mechanisms and Slopes

A combination of nucleation and growth

Structures

n = 5

Crack Nucleation

n = 10

Crack Growth

n = 3
Effect of Slope on Variability

Cumulative Probability

Equivalent Load

<table>
<thead>
<tr>
<th>n</th>
<th>COV</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.53</td>
</tr>
<tr>
<td>5</td>
<td>0.43</td>
</tr>
<tr>
<td>10</td>
<td>0.38</td>
</tr>
</tbody>
</table>
Loading History Variability

- Test Track
- Customer Service
Test Track Variability

40 test track laps of a motorhome

Cumulative Probability

Equivalent Load

COV 0.12
Customer Usage Variability

42 drivers / cars

Cumulative Probability

Equivalent Load

COV 0.32

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Variability in Environment

- Inclusions
- Pit depth
Inclusions That Initiated Cracks


COV = 0.27
Pits That Initiated Cracks

Crawford et. al. "The EIFS Distribution for Anodized and Pre-corroded 7010-T7651 under Constant Amplitude Loading"  

7010-T7651
Pre-corroded specimens

300 specimens
246 failed from pits
Pit Size Distribution

Mean = 230
COV = 0.32
Pit Depth Variability

Pits
12 Data Points
Median 24.37
COV 0.33

Cumulative Probability

Pit Depth, \( \mu \text{m} \)

Dolly, Lee, Wei, “The Effect of Pitting Corrosion on Fatigue Life”
Variability in Materials

- Tensile Strength
- Fracture Toughness
- Fatigue
  - Fatigue Strength
  - Fatigue Life
- Strain-Life
- Crack Growth
Tensile Strength - 1035 Steel

Mean = 602
COV = 0.045

Number of heats

Tensile Strength, MPa

500 550 600 650 700

Fracture Toughness

Cumulative Probability

26 Data Points
Median 76.7
COV 0.06

K_{lc}, Ksi \sqrt{\text{in}}

Fatigue Variability

![Fatigue Variability Graph]

- **Stress Amplitude** vs **Fatigue Life**
- **Fatigue life** and **Fatigue strength**

The graph shows the relationship between stress amplitude and fatigue life, with a focus on the distribution of fatigue lives and how stress amplitude affects fatigue strength.
Fatigue Life Variability

Production torsion bars
5160H steel

\( \mu_x = 123,000 \text{ cycles} \)
\( \text{COV} = 0.25 \)

\( \mu_x = 134,000 \text{ cycles} \)
\( \text{COV} = 0.27 \)

Statistical Variability of Fatigue Life

7075-T6 Specimens

Sinclair and Dolan, “Effect of Stress Amplitude on the Variability in Fatigue Life of 7075-T6 Aluminum Alloy” Transactions ASME, 1953
## COV vs Fatigue Life

<table>
<thead>
<tr>
<th>ΔS</th>
<th>$\bar{X}$</th>
<th>COV</th>
</tr>
</thead>
<tbody>
<tr>
<td>440</td>
<td>14,000</td>
<td>0.12</td>
</tr>
<tr>
<td>315</td>
<td>25,000</td>
<td>0.38</td>
</tr>
<tr>
<td>280</td>
<td>220,000</td>
<td>0.70</td>
</tr>
<tr>
<td>245</td>
<td>1,200,000</td>
<td>0.67</td>
</tr>
<tr>
<td>210</td>
<td>12,000,000</td>
<td>1.39</td>
</tr>
</tbody>
</table>
Variability in Fatigue Strength

\[ \frac{\Delta S}{2} = S^\prime_f(N_f)^b \quad b \approx -0.085 \]

\[ \text{COV} \quad C = \sqrt{\prod_{i=1}^{n}(1+C_{X_i}^2)^{a_i^2} - 1} \]

\[ C_{S^\prime_f} = \sqrt{(1 + 1.39^2)^{-.085}^2 - 1} = 0.088 \]
Strain Life Data for 950X Steel

378 Fatigue Tests
29 Data Sets
29 Individual Data Sets

\[ \sigma_f \]

Median 820
COV 0.25

Cumulative Probability

99.9%
99%
90%
80%
70%
60%
50%
40%
30%
20%
10%
5%
1%
0.1%
0.01%

Cumulative Probability

99.9%
99%
90%
80%
70%
60%
50%
40%
30%
20%
10%
5%
1%
0.1%
0.01%

Mean -0.09
COV 0.25
29 Individual Data Sets (continued)

Median 0.57
COV 1.15

Cumulative Probability

Mean -0.62
COV 0.23

Cumulative Probability
Input Data Simulation

\[
\frac{\Delta \varepsilon}{2} = \frac{\sigma_f \left( L, \mu_{\sigma_f}, \sigma_{\sigma_f} \right)}{E} (2N_f)^b(N, \mu_b, \sigma_b) + \varepsilon_f \left( L, \mu_{\varepsilon_f}, \sigma_{\varepsilon_f} \right) (2N_f)^c(N, \mu_b, \sigma_b)
\]
Simulation Results

![Graph showing simulation results for strain amplitude vs. fatigue life. The graph displays a trend where the strain amplitude decreases as the fatigue life increases, with data points and lines indicating variability.](image-url)
Correlation

\[ \rho = -0.828 \]
\[ \rho = -0.976 \]

\( \sigma'_f \) vs. \( b \) and \( \varepsilon'_f \) vs. \( c \)
Generating Correlated Data

\[ z_1 = \Phi(\ rand() ) \quad z_1 = N(0,1) \]

\[ z_2 = \Phi(\ rand() ) \]

\[ z_3 = z_1 \rho + z_2 \sqrt{1-\rho^2} \]

\[ \sigma_f' = \exp(\mu_{\ln\sigma_f} + \sigma_{\ln\sigma_f} z_1 ) \]

\[ b = \mu_b + \sigma_b z_3 \]
Curve Fitting

\[ \frac{\Delta \varepsilon}{2} = \frac{\sigma'_f}{E} (2N_f)^b + \varepsilon'_f (2N_f)^c \]

Assume a constant slope to get a distribution of properties.
Property Distribution

\( \sigma'_f \)

378 Data Points

Median 802 MPa

COV 0.12

b = -0.086

\( \varepsilon'_f \)

365 Data Points

Median 0.26

COV 0.42

c = -0.51
Strength Coefficient

Cumulative Probability

- 99.9%
- 99%
- 90%
- 50%
- 10%
- 1%
- 0.1%

- 365 Data Points
- Median: 1002 MPa
- COV: 0.14

\[ \rho = 0.863 \]
Crack Growth Data

Crack Growth Rate Data

Fatigue Lives

Cumulative Probability

99.9 %
99 %
90 %
50 %
10 %
1 %
0.1 %

300,000
68 Data Points
Median 257,000
COV 0.07

Crack Growth Rate

Stress intensity, MPa√m

10^{-5}
10^{-4}
10^{-3}
10^{-2}

5 10 50

Crack growth rate, mm/cycle
Crack Growth Properties

\[ \frac{da}{dN} = C \Delta K^m \]

Cumulative Probability

- Median: $5 \times 10^{-8}$
- COV: 0.44

Cumulative Probability

- Median: $3.13 \	imes 10^{-6}$
- COV: 0.06
Simulated Data

Cumulative Probability

- 99.9%
- 99%
- 90%
- 50%
- 10%
- 1%
- 0.1%

Data

Simulation

10^5 10^6 10^7
Beware of Correlated Variables

\[ N_f = \frac{a_f^{1-m/2} - a_i^{1-m/2}}{C \Delta S^m \pi^2 (1-m/2)} \]

\( N_f \) and \( C \) are linearly related and should have the same variability, but

\[ \text{COV}_{N_f} = 0.07 \]
\[ \text{COV}_C = 0.44 \]

because \( C \) and \( m \) are correlated.
Calculated Lives

Computed from C and m pairs

Experimental

Cumulative Probability

\[ N_f = \int_{a_o}^{a_f} \frac{da}{C(\Delta K)^m} \]
Manufacturing/Processing Variability

- Bolt Forces
- Surface Finish
- Drilled Holes
Variability in Bolt Force

- Force
- 200 Data Points
- Median 130
- COV 0.14

Preload force in bolts tightened to 350 Nm
Surface Roughness Variability

- 125 Data Points
- Median: 43
- COV: 0.10

Cumulative Probability

- Surface Finish
- Machined aluminum casting

- Cumulative Probability:
  - 99.9%
  - 99%
  - 99%
  - 90%
  - 50%
  - 10%
  - 1%
  - 0.1%
Drilled Holes

Fighter Spectrum
154 Data Points
Median 126,750
COV  0.22 in life
COV  0.07 in strength

Cumulative Probability

Analysis Uncertainty

- Miners Linear Damage rule
- Strain Life Analysis
Miners Rule

A safety factor of 10 in life would result in a 10% chance of failure

964 Tests
COV = 1.02

From Erwin Haibach “Betriebsfestigkeit”, Springer-Verlag, 2002
SAE Specimen

Fatigue Under Complex Loading: Analysis and Experiments, SAE AE6, 1977
Analysis Results

Strain-Life analysis of all test data

48 Data Points
COV 1.27
Material Variability

Cumulative Probability

Strain-Life back calculation of specimen lives

Analytical Life / Experimental Life
Modeling Uncertainty

Analysis Uncertainty $C_U = ?$

The variability in reproducing the original strain life data from the material constants is $C_M \sim 0.44$

$$COV \ C = \sqrt{\prod_{i=1}^{n} \left(1+C_{X_i}^2\right)^{a_i^2}} - 1$$

$$1 + C_U^2 = \frac{1 + C_{N_f}^2}{1 + C_M^2}$$

$C_U = 1.09$

90% of the time the analysis is within a factor of 3 !
99% of the time the analysis is within a factor of 10 !
Variability from Multiple Sources

\[
\text{COV} \quad C = \sqrt[\sum_{i=1}^{n} \left(1 + C_{X_i}^2 \right)^{a_i^2} - 1}
\]

Suppose we have 4 variables each with a COV = 0.1

The combined variability is COV = 0.29

Suppose we reduce the variability of one of the variables to 0.05

The combined variability is now COV = 0.27

If all of the COV’s are the same, it doesn’t do any good to reduce only one of them, you must reduce all of them!
Variability from Multiple Sources

\[ \text{COV} \quad C = \sqrt{\prod_{i=1}^{n} \left( 1 + C_{x_i}^2 \right) a_i^2} - 1 \]

Suppose we have 3 variables each with a COV = 0.1 and one with COV = 0.4.

The combined variability is COV = 0.65

Suppose we reduce the variability of these variables to 0.05

The combined variability is now COV = 0.60

If one of the COV’s is large, it doesn’t do any good to reduce the others, you must reduce the largest one!
Probabilistic Aspects of Fatigue